

## Polyakov Loop Correlations at Large N

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I describe a study of the two-point single-eigenvalue distribution correlation function of Polyakov loops in the confined phase of four dimensional  $SU(N)$  YM theory at large  $N$ . The reasons for the interest in this correlation function are explained. Analytical and numerical results are presented. Brief conclusions are drawn.

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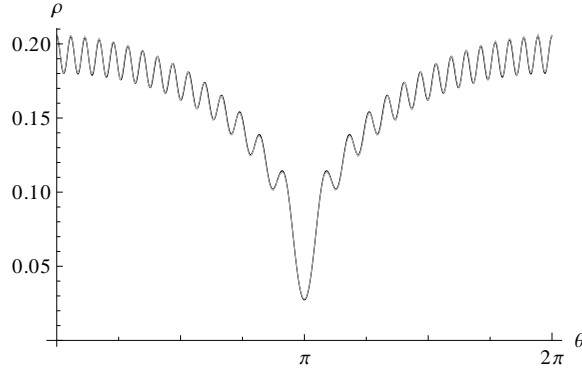
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## 1. Introduction

Quantum smeared loops [1] in pure  $SU(N)$  YM theory incur a qualitative change when they are dilated from a small to a large size. Controlling the crossover is a basic problem. At infinite  $N$ , the crossover collapses to a point, becoming a large- $N$  phase transition. Below the transition asymptotically free perturbation theory holds and above it a description by an effective string theory (EST) is valid. One would like to match these two ranges at the transition point and test that the matching works using lattice gauge theory. The problem one faces is that EST in its simplest form requires the loop to be smooth and contractible loops on the lattice have kinks. Non-contractible loops (Polyakov loops) do not have kinks but their expectation value is zero due to a  $Z(N)$  symmetry. So, the question is whether one can find a large  $N$  phase transition in the correlation function of two Polyakov loops at large  $N$ . Since this object is subleading in the large  $N$  expansion, the issue is subtle [2].

## 2. Large $N$ transition in contractible Wilson loops.

The single-eigenvalue distribution of smeared Wilson loops undergoes a “compactification” transition on the unit circle at  $N = \infty$  [3]. Below is an example of a  $6 \times 6$  smeared Wilson loop of size 0.6 Fermi at  $N = 29$ .



I would like to calculate approximately  $\sigma$  in units of  $\Lambda_{QCD}$  for  $N = \infty$  by matching EST (effective string theory) and PT (YM perturbation theory) at the transition point. This is a natural matching point: at  $N = \infty$  the parallel transporter round the loop does not reach the vicinity of the -1 element of the  $SU(N)$  group with probability one for smaller loops, while, for larger loops, the support of the eigenvalues of this parallel transporter is the entire group manifold. This indicates that the perturbative asymptotic expansion in the logarithm of loop size is a valid approximation for loops smaller than the transition size, but not for larger loops, where the compactness of the group manifold is detected and full exponentiation of the Lie algebra is necessary. EST is expected to be an asymptotic description for large loops with validity possibly extending to the entire regime of loop sizes exceeding the critical size.

Previous work [4] has led me to the conclusion that long distance behaviour is described by EST, but EST works in too limited a way for loops with kinks. EST requires smooth loops and one needs a situation where it works best. I also have to maintain the ability to test the matching procedure against Monte Carlo data. So, I need to work with smooth loops on the lattice.

Polyakov loops are the single available option. Hence, I look for an  $N = \infty$  transition associated with Polyakov loops within the low temperature phase.

### 3. Setup

I first define my notation. The Polyakov parallel transporter is denoted by

$$U_P(x) = \mathcal{P} e^{i \oint_{x_4} A_4(\vec{x}, \tau) d\tau}. \quad (3.1)$$

Here  $\vec{x}$  denotes the space component of the four-vector  $x$ . In the continuum limit, the quantum smeared  $U_P(x)$  is a matrix with operator valued entries which satisfies the same unitarity conditions a unitary c-number matrix would. The set of its eigenvalues  $e^{i\theta_k}$  is gauge invariant. The character of the parallel transporter in the irreducible representation  $R$  is given by  $P_R(\vec{x}) = \frac{1}{d_R} \chi_R(U_P(x))$ . It is independent of  $x_4$ . The two point correlation function of two Polyakov loops at two space points depends only on their spatial separation  $r$  and is denoted by  $G_R(r) = \langle P_R(0) P_R(r) \rangle$ .

The two point function is positive (the theta parameter in the YM action is set to zero) and its logarithm is a useful quantity:  $W_R(l, r) = \log G_R(r)$ , where  $l$  is the length of the compact direction and  $r$  the loop separation. As an example of possibly the strongest EST prediction consider the quantity  $\mathcal{F}_R(l)$ :

$$\mathcal{F}_R(l) = \lim_{r \rightarrow \infty} \partial^2 W_R(l, r) / \partial l \partial r \quad (3.2)$$

For  $1 \leq n \leq N-1$ , the “ $N$ -ality”, we consider  $\mathcal{F}_R(l) = \sigma_n \hat{F}_R(l\sqrt{\sigma_n})$ . This is the case where EST makes its strongest prediction in our context:

$$\hat{F}_R(x) = 1 + c_1/x^2 + c_2/x^4 + c_3/x^6 + \dots, \quad (3.3)$$

where the  $c_{1,2,3}$  are three universal, calculable numbers, independent of  $R$  and  $n$  [5]. Taking the large size limits in different ways typically produces weaker results.

### 4. 2D YM model

In the context of non-analyticities generated by taking  $N$  to infinity in the ’t Hooft prescription, previous work has shown that two dimensional YM theory provides a representative of the “universality class” associated with the large  $N$  transition. Therefore, I first study the eigenvalue-eigenvalue correlation for Polyakov loop matrices in 2D YM. Specifically, I compute a two point function of eigenvalue densities  $\rho^{(1)}(\theta; U) = \frac{2\pi}{N} \sum_{k=1}^N \delta_{2\pi}(\theta - \theta_k)$ .

One starts from the “propagator” [6]

$$Z_N(U_{P_1}, U_{P_2} | t) = \sum_R \chi_R(U_{P_1}) e^{-\frac{t}{2N} C_2(R)} \chi_{\bar{R}}(U_{P_2}), \quad (4.1)$$

intending to calculate

$$\langle \rho_1^{(1)}(\alpha) \rho_2^{(1)}(\beta) \rangle_c = \int dU_{P_1} dU_{P_2} \rho_1^{(1)}(\alpha) \rho_2^{(1)}(\beta) [Z_N(U_{P_1}, U_{P_2} | t) - 1] \quad (4.2)$$

This can be done using the character expansion [7] in terms of hook-type Young diagrams  $(p, q)$

$$\rho^{(1)}(\theta; U) = 1 + \frac{1}{2N} \lim_{\varepsilon \rightarrow 0^+} \sum_{p=0}^{N-1} \sum_{q=0}^{\infty} (-1)^p e^{-\varepsilon(p+q+1)} [e^{i(p+q+1)\theta} \chi_{(p,q)}(U) + e^{-i(p+q+1)\theta} \chi_{(\overline{p,q})}(U)]. \quad (4.3)$$

For simplicity, I will restrict myself to odd  $N$ . Using  $C(p, q) = (p + q + 1)(N - \frac{p+q+1}{N} + q - p)$ , I obtained

$$\langle \rho_1^{(1)}(\alpha) \rho_2^{(1)}(\beta) \rangle_c = \frac{1}{N^2} \sum_{p=0}^{N-1} \sum_{q=0}^{\infty} (-1)^p e^{-\frac{t}{2N} C(p,q)} \cos[(p + q + 1)(\alpha - \beta)] \quad (4.4)$$

Taking the large  $N$  limit gives:

$$N^2 \langle \rho_1^{(1)}(\alpha) \rho_2^{(1)}(\beta) \rangle \sim \Re \sqrt{\frac{N}{t}} u e^{-\frac{t}{2} + \frac{t}{2N^2}} \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{N}{2t} x^2 + \frac{1}{2t} x^2} \frac{1 + u^N e^{-N(x + \frac{t}{2}) + \frac{t}{2}}}{1 + u e^{-x - \frac{t}{2} + \frac{t}{2N}}} \frac{1}{1 - u e^{-\frac{t}{2}}}, \quad (4.5)$$

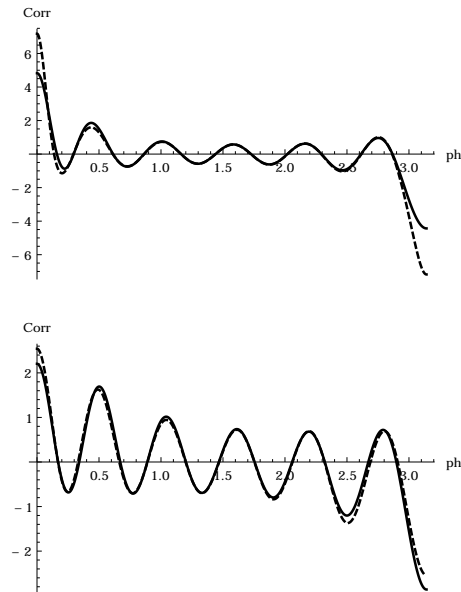
where  $u = \exp[i(\alpha - \beta)]$ .

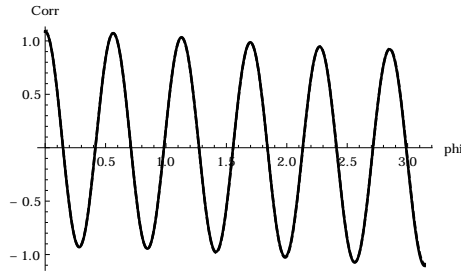
The answer consists of the sum of a rapidly oscillating piece and a non-oscillating piece

$$\frac{1}{2} \frac{\sinh \frac{t}{2} \cos \phi}{\sinh^2 \frac{t}{2} + \sin^2 \phi}, \quad (4.6)$$

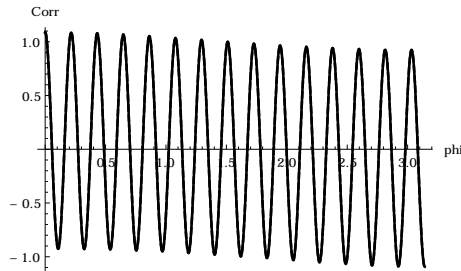
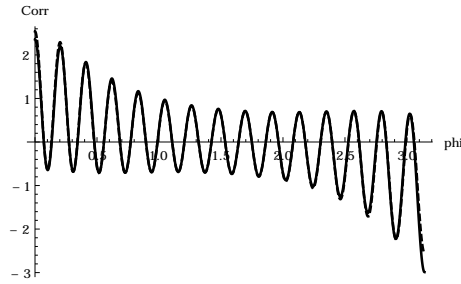
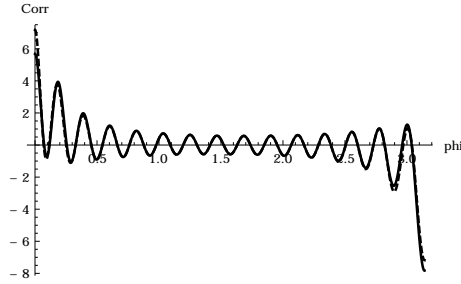
where  $\phi = \alpha - \beta$ . The expression differs from the universal form for random hermitian matrix models [8], likely because of the absence of the potential term of the latter. There is no large  $N$  transition separating regimes of small  $t$  and large  $t$ . The approximate large  $N$  formula is compared with the exact finite  $N$  formula below with the solid line showing the exact result. One sees that the approximate large  $N$  expression deteriorates when  $N$  decreases, when  $\phi \approx k\pi, k \in \mathbb{Z}$  and when  $t$  is small relative to 1.

$N = 11, t = 0.3, 1, 5$  from top to bottom:





$N = 29, t = 0.3, 1, 5$  top to bottom:



We see that the large  $N$  expression I derived analytically checks against finite  $N$  expressions evaluated numerically.

## 5. 4D results

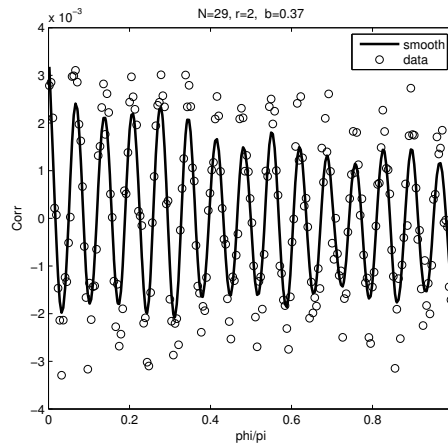
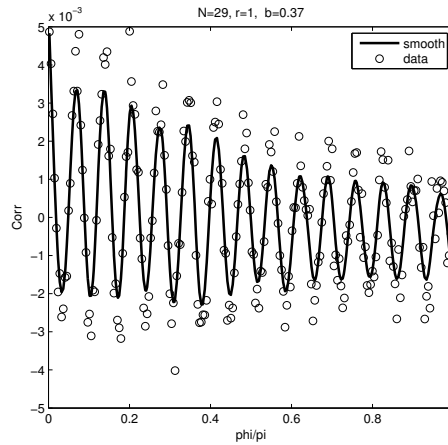
For finite  $N$ , there is no reason for  $\rho^{(2)}$  to depend only on the angle difference since the  $Z(N)$  symmetry only provides invariance under simultaneous shifts of  $\alpha$  and  $\beta$  by  $2\pi k/N$ . Initial simulations were done collecting two dimensional histograms in the  $\alpha, \beta$  plane. It was found that

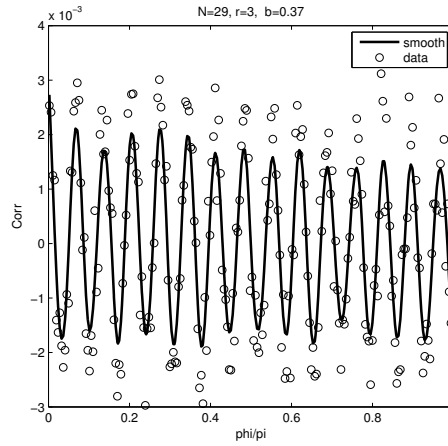
within practical numerical accuracy collapsing the histograms along constant  $\alpha - \beta$  lines did not loose any information. This means that we may as well redefine  $\rho^{(2)}$ :

$$\rho^{(2)}(\alpha - \beta) = \frac{N}{2\pi} \int_{-\pi/N}^{\pi/N} d\theta \langle \rho_1^{(1)}(\alpha + \theta) \rho_2^{(1)}(\beta + \theta) \rangle_c \quad (5.1)$$

producing a  $\rho^{(2)}$  depending only on the angle difference on account of the  $Z(N)$  symmetry.

An example of the outcome of a Monte Carlo simulation in 4D is shown below. In addition to raw data, I show a smoothed curve obtained by a cubic spline smoothing method. The method of smoothing consists of a minimization of a weighted combination of an average of the curve curvature and deviation from the data. The smoothing procedure is quite ad-hoc, and only serves to produce curves to guide the eye.





These results were obtained for  $N = 29$  and rescaled 't Hooft coupling  $b(\equiv \frac{\beta}{2N^2}) = 0.370$  at separation  $r = 1, 2, 3$  in lattice units from top to bottom. Only half of the angular range is shown.

Qualitatively, the curves resemble their two dimensional counterparts, but the noise is large. The results indicate no large  $N$  phase transition in this observable in 4D. I have not ruled out that the redefinition in eq. (5.1) hid a transition. It would be numerically expensive to do this.

## 6. Conclusions and Outlook

There is no large  $N$  phase transition for large enough Polyakov loops as their separation is varied. To get a large  $N$  transition one would have to shrink the compact direction, while maintaining the system in the confined phase. This phase would be metastable. This may be possible using quenching techniques and would be of theoretical interest also in another respect [9].

Other observables, involving the analogue of the 2D YM “vertex”, and which combine different windings might be of interest and could potentially provide better candidates for observables undergoing large  $N$  phase transitions. For more details I refer to [2].

## References

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